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# NUMERICAL METHODS FOR $H_2$ RELATED PROBLEMS

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## Abstract

Recent results have shown that several  $H_2$  and  $H_2$ -related problems can be formulated as convex programs with a *finite* number of variables. We present an interior point algorithm for the solution of these convex programs and illustrate its application with the standard LQR design.

## 1. Introduction

It has been shown recently that a number of  $H_2$  and  $H_2$ -related problems can be formulated as convex programs with a finite number of variables — quadratic stabilization [1], mixed  $H_2/H_\infty$  and multicriterion LQG problems (see [2] and references therein). The common idea underlying these results is that though the original problem is not convex, a clever change of variables [3] makes it convex.

In this paper, we present a systematic procedure for transforming the convex programs resulting from  $H_2$ -related problems above into optimization over Affine Matrix Inequalities. We then present a simple interior point method for their solution. Though our presentation is through the simple LQR design example, the techniques readily extend to the more complicated problems cited above.

## 2. The LQR problem

Consider the linear time invariant system described by the state equations

$$\begin{aligned} \dot{x} &= Ax + Bu + w \\ z &= \begin{bmatrix} R^{\frac{1}{2}} & 0 \\ 0 & Q^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} \end{aligned} \quad (1)$$

where  $u$  is the control input,  $w$  is unit intensity white noise and  $z$  is the output signal of interest. The LQR problem is to design a feedback controller from the state  $x$  to the control input  $u$  which minimizes the

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$H_2$  norm between  $w$  and  $z$  [4]. It is known that the optimal feedback law is a constant state feedback  $u = -Kx$  and optimizes the following program:

$$\min_{P,K} \text{Tr}(QP) + \text{Tr}(R^{1/2}KPK^TR^{1/2})$$

subject to

$$(A - BK)P + P(A - BK)^T + I < 0 \text{ and } P = P^T > 0.$$

By defining a new quantity  $Y = KP$ , the above problem can be written as

$$\min_{P,Y} \text{Tr}(QP) + \text{Tr}(R^{1/2}YP^{-1}Y^TR^{1/2}) \quad (2)$$

subject to

$$AP + PA^T - BY - Y^TB^T + I < 0 \text{ and } P = P^T > 0,$$

which is a convex program (see, for example, [1], [2]).

## 3. Transformation to Optimization over AMI's

The general structure of the convex program (2) is

$$\min_{(P,Y) \in \mathcal{A}} J(P, Y)$$

where  $\mathcal{A}$  is some convex constraint set, and  $J$  a performance index. We show how this can be transformed into the problem

$$\min_{C(\gamma,Z) > 0} \gamma \quad (3)$$

where  $(\gamma, Z)$  is a new set of variables and  $C$  is symmetric and an affine matrix function of  $(\gamma, Z)$ . The inequality  $C(z) > 0$  is called an *Affine Matrix Inequality* (AMI).

### Example: The LQR problem

The objective function of program (2) consists of the sum of two terms. It is easily shown that the second term

$$\Phi(P, Y) = \text{Tr}(R^{1/2}YP^{-1}Y^TR^{1/2})$$

can be expressed as

$$\Phi(P, Y) = \min(\text{Tr}(X))$$

$$\begin{bmatrix} X & R^{1/2}Y \\ Y^T R^{1/2} & P \end{bmatrix} > 0.$$

Then let

$$C_1(\gamma, P, Y, X) := -\text{Tr}(QP) - \text{Tr}(X) + \gamma,$$

$$C_2(\gamma, P, Y, X) := -AP - PA^T + BY + Y^T B^T - I,$$

$$C_3(\gamma, P, Y, X) := \begin{bmatrix} X & R^{1/2}Y \\ Y^T R^{1/2} & P \end{bmatrix},$$

$$C(\gamma, P, Y, X) := \text{diag}(C_1, C_2, C_3).$$

The optimization problem (2) can now be written

$$\begin{aligned} & \text{minimize } \gamma \\ & C(\gamma, P, Y, X) > 0 \end{aligned} \quad (4)$$

which indeed is of the form (3).

### Well-Posedness

We say that the convex program (3) is well-posed if for every real  $\gamma$  the set  $\{Z \mid C(\gamma, Z) > 0\}$  is compact or empty. We observe the following without proof:

**Proposition 1:** *The program (4) corresponding to the LQR problem is well-posed if  $R$  is positive definite,  $(A, B)$  is controllable and  $(Q, A)$  is observable.*

Under similar assumptions, all the other problems cited in the introduction enjoy the same property.

### 4. Computational aspects

Problem (3) is a convex non-differentiable optimization program, and the ellipsoid algorithm or Kelley's cutting-plane algorithm [4] may be used to solve it. Recently, the work of Nemirovski *et al.* has led to the development of interior point algorithms based on the notion of the *analytic center* for a set of convex constraints [5]; these algorithms seem to hold great promise.

We will describe one such interior point algorithm, called the *method of centers*. Given an initial feasible point  $(\gamma_u, Z_u)$  for constraint  $C$  in program (3), and a desired absolute accuracy  $\epsilon$  on the optimum, the algorithm is as follows:

**while**  $\gamma_u - \gamma_l > \epsilon$ ,  
 $\gamma_0 := \gamma_u + \epsilon$ ,  
 $(\gamma^*, Z^*) := \text{a\_center}(\text{diag}(C, \gamma_0 - \gamma) > 0)$ ,  
 $\gamma_u := \gamma^*$ ,

$Z_u := Z^*$ ,

Compute a lower bound  $\gamma_l$ .

**end**

**Remark 1:** An initial lower bound  $\gamma_l$  can be chosen to be 0.

**Remark 2:** Computing the analytic center of a convex bounded set ( $\text{a\_center}(\text{diag}(C, \gamma_0 - \gamma) > 0)$ ) needs an initial point interior to the constraint.  $(\gamma_u, Z_u)$  is such a point.

**Remark 3:** The lower bound  $\gamma_l$  is computed from  $\gamma^*$  and the Hessian of the barrier function of the constraint expressed at  $(\gamma^*, Z^*)$ . For more information, see [5], [6] and also [7] (these proceedings).

### 5. Conclusion

Through the LQR example, we have outlined a systematic procedure for transforming convex optimization programs arising from  $H_2$ -related problems into optimization over AMI's. We have also briefly described a simple interior point method for their solution. Our procedure easily applies to problems in [1] and [2], and more generally to many quadratic Lyapunov function shaping problems.

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